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Visualization of the Contact Force Solution Space for Multi-Limbed Robots

A new analytical method for determining, describing, and visualizing the solution space for the contact force distribution of multi-limbed robots with three feet in contact with the environment in three-dimensional space is presented. The foot contact forces are first resolved into strategically defined foot contact force components to decouple them, and then the static equilibrium equations are applied. Using the friction cone equation at each foot contact point, the problem is then transformed into a geometrical one. Using geometric properties of the friction cones and by simple manipulation of their conic sections, the entire solution space which satisfies the static equilibrium and friction constraints at each contact point can be found. Two representation schemes, the "force space graph" and the "solution volume representation," are developed for describing and visualizing the solution space which gives an intuitive visual map of how well the solution space is formed for the given conditions of the system. [DOI: 10.1115/1.2118732]

1 Introduction

This paper addresses the determination, representation, and visualization of the contact force solution space of a multi-limbed mobile robotic system with three feet in contact with its environment. Since the limbs of the robot have enough joints and are actively controlled, the contact forces at the feet need to be explicitly chosen to ensure that the robot does not lose its balance and does not slip at a foot. One of the major difficulties associated with finding the force distribution solution has been the indeterminate nature of the problem. Any system with three or more contact points makes it an underspecified system involving redundancy since there are more unknowns than the number of equations. The other major difficulty associated with the problem of multi-contact force distribution has been the nonlinear nature of the three-dimensional friction cone model.

Instead of finding just a single contact force solution through optimization methods as is the case for most previous work on this subject, the presented method describes all the possible solutions (solution space) for the contact force distribution for a statically stable body under friction constraints. The optimal contact force solution can then be chosen in this solution space which maximizes the objectives given by the chosen optimization criteria. This two-step approach allows one to have more options and freedom in choosing the final solution and to satisfy other special conditions that might be considered at that instant. This paper presents the method for finding the solution space as the first step of finding the optimal contact force distribution. The second step of choosing the optimal solution in this solution space was presented in [1–3].

2 Background and Previous Work

2.1 The Multi-Contact Force Distribution Problem. Since a multi-limbed mobile robotic system has multiple contacts with its environment, the contact forces required to support it and those required by its tasks are indeterminate. This problem also arises in the study of multi-fingered robotic hand grasping [4,5] and multiple cooperating robot arms with closed kinematic chains [6]. If

the robot is supported by passively constrained legs, the statically indeterminate force and moments at the foot contact points would have to be determined by examining the flexibility of the contact surface and the limbs. However, since the limbs of the robot have enough joints and they are actively controlled, the components of the contact forces need to be explicitly chosen which ensures that the robot does not lose its balance and does not slip at a foot. Most investigations for solving this multi-contact force distribution problem so far have been done in the context of multi-fingered robotic hands or multi-limbed walking machines and solutions have been attempted by using many different optimization methods.

For a multi-limbed robot to maintain a stable position on a steep incline or on rough terrain, there are two key issues that need to be considered: one is losing balance due to external forces such as gravity, and the other is the slipping between the feet and the surface. The first has to do with the force and moment balance of the robot and the second has to do with the friction between each foot and the surface. In other words, all the forces acting on the robot including the foot contact forces should satisfy the static equilibrium equations while at the same time, each foot contact force should be in the friction cone at the contact point to avoid slipping.

One major difficulty associated with this topic has been the indeterminate nature of the problem. There are only six equations for static equilibrium (three for the force balance and three for the moment balance in three dimensions) while the number of unknown forces is three times the number of contact points (assuming a frictional point contact), thus any system with three or more contact points makes it an underspecified system involving redundancy. Also, the nonlinear nature of the three-dimensional friction cone model is another major difficulty associated with the multicontact force distribution problem.

We define the multi-contact force distribution problem for a multi-limbed mobile robot with three feet contact as follows: Given the locations of the three foot contact points on the surface and their corresponding friction models, and the known external load on the robot body, determine all the possible contact forces that will balance the load (static equilibrium) and preclude slip at any contact points (friction constraints).

2.2 Previous Work. The force distribution problem for multilimbed robotic vehicles is a statically indeterminate problem for-

Contributed by Mechanisms and Robotics Committee of ASME for publication in the JOURNAL OF MECHANICAL DESIGN. Manuscript received June 28, 2004; final manuscript received June 24, 2005. Assoc. Editor: Hashem Ashrafiuon.

mulated by a set of equality constraints (static force moment equilibrium equations) and a set of nonlinear inequality constraints (friction constraints). The goal is to find the solution that satisfies these constraints and maximizes the objectives given by the optimization criteria.

Most previous work in this area proposed using the pseudoinverse [7,8] together with various optimization techniques for finding a solution for a problem formulated as a constrained, optimization problem using specific objective functions [9-17]. The problem is then solved by linearizing the friction cone constraints first and then applying various linear programming techniques. These approaches were also used for multi-fingered robotic hand grasping [14,18,19] and multiple robot manipulators working together. However, these methods were often too slow for real time computation and were limited in many ways. There were also attempts to find a suboptimal solution more quickly to make it fast enough for real time computation [19,20]. More recent work [21] includes analyzing and synthesizing grasps in a constraint of combined elasticity and geometric compatibility. In addition to the force equilibrium condition Ref. [22] presents a searching method for finding a form-closures grasp, and Ref. [23] presents a numerical test for force and form closure for a given grasp.

Unlike other methods previously developed where a single solution to a constrained optimization problem is immediately found from maximizing an objective function(s) under constraints, Hong and Cipra [24] introduced a two-step approach in solving the multi-contact force distribution problem. First, the entire solution space that satisfies all constraints is found (finding the solution space), and then a solution in that space which will give the best results for the objectives given by the chosen optimization criteria is chosen as its optimal solution (choosing the optimal solution). Finding the description of the entire solution space first provides an intuitive visual map of how well the solution space is formed for the given conditions of the system. This is very important and can be useful to the higher-level motion planner for deciding on potential foot placement locations on the surface or for choosing the internal configuration of the robot (posture) when moving. Choosing a solution in the solution space will give insight into the quality of that chosen solution and provide a measure of robustness against disturbances, thus will allow us to choose the "best" solution for the situation.

In [24], the method developed was applicable only to a climbing tethered mobile robot with a single cable and two feet contact. In this paper we present a more general method for representing and visualizing the contact force solution space for multi-limbed mobile robots with three feet contact as the first step in finding the optimal contact force distribution solution. The methods for both cases are similar in concept, but the method presented in this paper for the three feet contact case is more general and thus can also be used for finding the optimal force distribution for the one cable-two feet contact case as well.

3 Determining, Describing, and Visualizing the Solution Space

3.1 Overview of the Method. The overall strategy for finding and describing the contact force solution space is similar to that for the "one cable-two feet contact case" [24]; however, the entire solution space which satisfies the static equilibrium and friction constraints at each contact point is described in terms of three parameters instead of one, and thus the strategy for finding the description of the solution space is more complex.

The foot contact forces are first resolved into strategically defined foot contact force components to decouple them for simplifying the solution process, and then the static equilibrium equations are applied to find certain contact force components and the relationship between the others. Using the friction cone equation at each foot contact point and the known contact force components, the problem is transformed into a geometrical one to find the ranges of contact forces and the relationship between them that satisfy the friction constraint. Using geometric properties of the friction cones and by simple manipulation of their conic sections, the entire solution space which satisfies the static equilibrium and friction constraints at each contact point can be found.

The "force space graph" and the "solution volume representation" schemes are developed for describing and visualizing the solution space which gives an intuitive visual map of how well the solution space is formed for the given conditions of the system. These can be used as tools for choosing the final optimal solution in its solution space as presented in [1-3].

3.2 Assumptions and the Mobile Robotic System. The multi-limbed mobile robotic system under consideration is assumed to move in a quasi-static manner, by stably supporting itself against the environment surfaces using its three feet, moving a free limb to a new position and setting it down while lifting a foot that was in contact with the surface, and then slowly changing its internal configuration to a new posture. The "three feet contact" implies that there are only three feet in contact with the surface at a given time, and thus does not mean that the robot only has three feet.

The following describes assumptions made for the robotic system under consideration:

- (a) We assume that the mechanism of each leg has enough degrees of freedom and the joints have large enough joint torque limits such that each foot can exert a force to the surface in any direction at any magnitude.
- (b) We treat all external forces and moments acting on the robot (except for those from the feet contact points) as known forces including gravity, wind loads, or wrenches acting on the robot during manipulation tasks, etc. and represent them by a single force and a moment acting on a point on the body.
- (c) We assume that the contact at the foot follows the "point contact with friction" model (Salisbury [5]) where only a normal force and a tangential force are acting on the contact point, as opposed to the "soft finger contact" model [25] where the contact point is also subjected to a moment about the normal.
- (d) We assume the robot moves in a quasi-static manner.
- (e) We do not consider the degenerate case of when all three contact points are located on a straight line.

Only the computation of the possible contact forces is of concern in this work, and the effects of leg stiffness and the problem of controlling the force and compliance are not in the scope of this method.

3.3 The Unit Contact Force Component Vectors and the Force System. We first define vectors and coordinate frames as shown in Fig. 1. The *O-XYZ* body coordinate frame is attached to an arbitrary point at an arbitrary orientation on the body; however, it is convenient to attach it at the center of gravity of the body. The vectors r_{Ci} (*i*=1,2,3) are the position vectors for the three foot contact points C_1 , C_2 , and C_3 from the origin of the body coordinate frame is defined using three of the following four unit contact force component vectors e_{α} , e_{β} , e_{γ} and e_{δ} .

The unit direction vector pointing from C_1 to C_2 is defined as the unit contact force component vector e_{α} , the unit direction vector pointing from C_2 to C_3 is defined as the unit contact force component vector e_{β} , and the unit direction vector pointing from C_3 to C_1 is defined as the unit contact force component vector e_{γ} . That is,

$$e_{\alpha} = (r_{C2} - r_{C1})/||r_{C2} - r_{C1}|| \tag{1}$$

$$e_{\beta} = (r_{C3} - r_{C2}) / \|r_{C3} - r_{C2}\|$$
(2)

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Fig. 1 The coordinate system

$$e_{\gamma} = (r_{C1} - r_{C3}) / \|r_{C1} - r_{C3}\|$$
(3)

The unit contact force component vector e_{δ} is defined to be perpendicular to the foot contact plane (the plane defined by the three contact points C_1 , C_2 and C_3) following the right hand coordinate rule as shown in Fig. 1. Thus,

$$e_{\delta} = (e_{\alpha} \times e_{\beta}) / \|e_{\alpha} \times e_{\beta}\| \tag{4}$$

These four unit contact force component vectors e_{α} , e_{β} , e_{γ} , and e_{δ} are then arranged as shown in Fig. 1 to form the three foot contact coordinate frames used for describing the foot contact forces at each foot contact point. Note that these are not orthogonal coordinate frames. Resolving the foot contact forces into these newly defined contact force components, instead of using the usual X-Y-Z Cartesian coordinates, decouples the foot contact forces to enable finding certain force components directly, and helps express the relationship of these force components between each contact point.

At each contact point C_1 , C_2 , and C_3 , the contact surface normal directions are represented by the unit surface normal direction vectors u_{N1} , u_{N2} , and u_{N3} , and the friction cones at these contact points are defined using these unit surface normal direction vectors together with their corresponding friction coefficients μ_{C1} , μ_{C2} , and μ_{C3} as shown in Fig. 2.

The system of all the external forces and moments due to gravity or any wrenches from interaction with the environment, etc. is represented by a known force \bar{F}_O and a known moment \bar{M}_O acting on the origin of the body coordinate frame O. The foot contact forces \overline{F}_{C1} , \overline{F}_{C2} , and \overline{F}_{C3} at points C_1 , C_2 , and C_3 are shown in their components $F_{C1\gamma}$, $F_{C1\alpha}$, $F_{C1\delta}$ and $F_{C2\alpha}$, $F_{C2\beta}$, $F_{C2\delta}$ and $F_{C3\gamma}$, $F_{C3\gamma}$, $F_{C3\delta}$, respectively. For the robot to be in static balance, all of these forces shown in

Fig. 2 must satisfy the static force moment equilibrium equations. In addition to this, the foot contact forces \overline{F}_{C1} , \overline{F}_{C2} , and \overline{F}_{C3} must be in their friction cones for the feet not to slip.

3.4 Applying the Static Equilibrium Equations. To find the solution space for the force distribution problem is to find all possible combinations of the foot contact forces at each foot contact point that satisfy the static force moment balance equations and the friction constraints at each foot contact point. Since there are only six equations for the static equilibrium condition but there are nine unknown force components (three components for each of the three contact points) not all force components can be directly solved for. However, by resolving each foot contact force

into the strategically defined foot contact coordinate frame components, we can directly find three of them $(F_{C1\delta}, F_{C2\delta}, \text{and } F_{C3\delta})$ using three of the static equilibrium equations.

 $F_{C1\delta}$, the e_{δ} component foot contact force at C_1 , can be found by summing the moments about e_{β} going through C_2 and C_3 , and solving

$$e_{\beta} \cdot ((r_{C1} - r_{C2}) \times \bar{F}_{C1\delta} + (-r_{C2}) \times \bar{F}_{O} + \bar{M}_{O}) = 0$$
 (5)

 $F_{C2\delta}$, the e_{δ} component foot contact force at C_2 , can be found by summing the moments about e_{γ} going through C_3 and C_1 , and solving

$$e_{\gamma} \cdot ((r_{C2} - r_{C3}) \times \bar{F}_{C2\delta} + (-r_{C3}) \times \bar{F}_{O} + \bar{M}_{O}) = 0$$
(6)

 $F_{C3\delta}$, the e_{δ} component foot contact force at C_3 , can be found by summing the moments about e_{α} going through C_1 and C_2 , and solving

$$e_{\alpha} \cdot ((r_{C3} - r_{C1}) \times \bar{F}_{C3\delta} + (-r_{C1}) \times \bar{F}_{O} + \bar{M}_{O}) = 0$$
(7)

The remaining six force components $F_{C1\alpha}$, $F_{C2\alpha}$, $F_{C2\beta}$, $F_{C3\beta}$, $F_{C3\gamma}$ and $F_{C1\gamma}$ cannot be found explicitly at this time due to the indeterminate nature. However, by summing the moments about e_{δ} at each three contact points C_3 , C_1 , and C_2 we can obtain three relationship equations between them as

$$e_{\delta} \cdot ((r_{C1} - r_{C3}) \times (\bar{F}_{C1\alpha} + \bar{F}_{C2\alpha}) + (-r_{C3}) \times \bar{F}_{O} + \bar{M}_{O}) = 0 \quad (8)$$

which is in the form of $F_{C1\alpha} + F_{C2\alpha} = \text{Constant}_1$, and

$$e_{\delta} \cdot ((r_{C2} - r_{C1}) \times (\bar{F}_{C2\beta} + \bar{F}_{C3\beta}) + (-r_{C1}) \times \bar{F}_{O} + \bar{M}_{O}) = 0 \quad (9)$$

which is in the form of $F_{C2\beta}+F_{C3\beta}=\text{Constant}_2$ and

$$e_{\delta} \cdot ((r_{C3} - r_{C2}) \times (\bar{F}_{C3\gamma} + \bar{F}_{C1\gamma}) + (-r_{C2}) \times \bar{F}_{O} + \bar{M}_{O}) = 0$$
(10)

which is in the form of $F_{C3\gamma}+F_{C1\gamma}=\text{Constant}_3$. Note that the six equations used are not in the form of three sums of the forces equations and three sums of the moments equations but are in the form of six sums of the moments equations as shown. This is still valid as long as the six equations used form a linearly independent set. Also, the foot contact forces have to be in compression since unlike a cable [24] the feet cannot exert force to the ground through tension. Future work will include extending this method for dry adhesive feet (gecko feet [26]) which can apply forces to the ground in all directions.

To illustrate the procedure, a robotic system with three feet in contact with the surface will be used as an example. The param-

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Fig. 3 The example system

eters used to define this example system and the numbers presented here as results could assume any units as long as they have the correct dimensions corresponding to its type (force or length). Position vectors for the three foot contact points C_1 , C_2 , and C_3 expressed in, and from the origin of the body coordinate frame are given as

$$\overline{r}_{C1} = \{4.0, 4.0, -3.9\}$$

$$\overline{r}_{C2} = \{-4.0, 4.1, -4.0\}$$
 (11)

 $\overline{r}_{C3} = \{0.0, -4.0, -4.1\}$

and at each foot contact point C_1 , C_2 , and C_3 , the foot contact coordinate frames are set following the rules defined using the four unit contact force component vectors e_{α} , e_{β} , e_{γ} , and e_{δ} as shown to scale in Fig. 3. The unit surface normal direction vectors u_{N1} , u_{N2} , and u_{N3} are given as

$$\overline{u}_{N1} = \{-0.577, -0.577, 0.577\}$$

$$\overline{u}_{N2} = \{0.667, -0.333, 0.667\}$$
 (12)

$$\overline{u}_{N3} = \{0.000, 0.707, 0.707\}$$

and their corresponding friction coefficients μ_{C1} , μ_{C2} , and μ_{C3} are

$$\mu_{C1} = 0.2, \quad \mu_{C2} = 0.32, \quad \mu_{C3} = 0.15$$
 (13)

The known external force \overline{F}_O and the known external moment \overline{M}_O expressed in the body coordinate frame and acting on the origin O are

$$\bar{F}_{O} = \{0.0, 0.1, -8.0\}, \quad \bar{M}_{O} = \{2.0, 0.1, 0.0\}$$
 (14)

This example system is represented to scale in Fig. 3 with the friction cones shown at each contact point.

Now we use this example system to illustrate the process of applying the static equilibrium constraints. First, we can directly solve for the three force components using the strategy illustrated above. The $F_{C1\delta}$, $F_{C2\delta}$, and $F_{C3\delta}$ component forces by summing the moments about e_{α} , e_{β} , and e_{γ} , and their magnitudes are then

$$F_{C1\delta} = 1.901, \quad F_{C2\delta} = 1.876, \quad F_{C3\delta} = 4.223$$
 (15)

The relationship between the force components $F_{C1\alpha}$ and $F_{C2\alpha}$, $F_{C2\beta}$ and $F_{C3\beta}$, and $F_{C3\gamma}$ and $F_{C1\gamma}$ are obtained by summing the moments about e_{δ} at the three contact points C_1 , C_2 , and C_3 as

$$F_{C1\alpha} + F_{C2\alpha} = -0.048 \tag{16}$$

$$F_{C2\beta} + F_{C3\beta} = 0.033 \tag{17}$$

$$F_{C3\gamma} + F_{C1\gamma} = 0.088 \tag{18}$$

Using the six static force moment balance equations, we have now found three force components and three relationships between the other six unknown force components. These values and relationships satisfy the static equilibrium conditions whether or not they satisfy the friction constraints.

3.5 Applying the Friction Cone Constraints. For the feet not to slip, the contact force at each contact point should also satisfy the friction constraints. Using the three force components found and the three relationship equations together with the friction constraints at each contact point, the remaining foot contact force components can be expressed in terms of three parameters with given ranges. The value of these three parameters can be chosen later to optimize the foot contact forces using the chosen optimization criteria.

Using the unit surface normal direction vectors u_{N1} , u_{N2} , and u_{N3} together with their corresponding friction coefficients μ_{C1} , μ_{C2} , and μ_{C3} , the equation for the friction cones at these contact points are defined in terms of their foot contact coordinate axes. Using these friction cone equations, the friction constraints at each foot contact point can then be represented as

$$\operatorname{Cone}_{C1}(F_{C1y}, F_{C1a}, F_{C1b}) \le 0 \tag{19}$$

$$\operatorname{Cone}_{C2}(F_{C2\alpha}, F_{C2\beta}, F_{C2\delta}) \le 0 \tag{20}$$

$$\operatorname{Cone}_{C3}(F_{C3\beta}, F_{C3\gamma}, F_{C3\delta}) \leq 0 \tag{21}$$

meaning that each foot contact force must be inside their respective friction cone.

Substituting the three known force component values $(F_{C1\delta}, F_{C2\delta})$ and $F_{C3\delta}$ in these quadratic friction constraint inequality equations, we now have three quadratic inequality equations with two variables each $(F_{C1\gamma}, F_{C1\alpha})$ and $F_{C2\alpha}, F_{C2\beta}$ and $F_{C3\beta}, F_{C3\gamma})$ as

$$E_{C1}^{\#}(F_{C1\gamma}, F_{C1\alpha}) \le 0 \tag{22}$$

$$E_{C2}^{\#}(F_{C2\alpha}, F_{C2\beta}) \le 0 \tag{23}$$

$$E_{C3}^{\#}(F_{C3\beta}, F_{C3\gamma}) \le 0 \tag{24}$$

Geometrically, this strategy is illustrated in Fig. 4 for contact point C_1 . The cross section of the friction cone [Fig. 4(*a*)] sliced by the force plane [Fig. 4(*b*)], which is defined by the known force component $F_{C1\delta}$ and is parallel to the e_{γ} - e_{α} plane, is shown in Fig. 4(*c*). This cross section area as shown in Fig. 4(*d*) represents the $F_{C1\gamma}$ - $F_{C1\alpha}$ plane region that satisfies the friction constraint for that contact point. Note that this cross section is a conic section which can be either an ellipse, parabola, hyperbola, circle, line, two non-parallel lines, or a point depending on how the force plane slices the friction cone (Fig. 5). This geometric property will be utilized later in the process of finding the optimal solution as presented in [1–3].

Now we have three quadratic friction constraint inequality equations, with a total of six force component variables $(F_{C1\gamma}, F_{C1\alpha}, F_{C2\alpha}, F_{C2\beta}, F_{C3\beta}, \text{ and } F_{C3\gamma})$ in the form of three conic section inequality equations. By applying the three linear relationships for $F_{C1\alpha}$ and $F_{C2\alpha}$, for $F_{C2\beta}$ and $F_{C3\beta}$, and for $F_{C3\gamma}$ and $F_{C1\gamma}$ from Eqs. (8)–(10), we can represent the solution space using only three force component variables. Substituting one variable each $(F_{C1\gamma}, F_{C2\alpha}, \text{ and } F_{C3\beta})$ in the three conic section in-

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Fig. 4 Geometric interpretation of finding the range of $F_{C1\gamma}$ and $F_{C1\alpha}$

equality Eqs. (22)–(24) with those from the three linear relationships and eliminating them, we now have a new set of three conic section inequality equations with two variables each, but a total of only three force component variables $(F_{C1\alpha}, F_{C2\beta}, F_{C3\gamma})$ as

$$E_{C1}(F_{C3\gamma}, F_{C1\alpha}) \le 0 \tag{25}$$

$$E_{C2}(F_{C1\alpha}, F_{C2\beta}) \le 0 \tag{26}$$



Fig. 5 Examples of conic sections

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$$E_{C3}(F_{C2\beta}, F_{C3\gamma}) \le 0 \tag{27}$$

This process can be better understood geometrically as flipping and shifting each conic section about an axis (a simple linear transformation) as will be shown in the following example.

Now any set of force components ($F_{C1\alpha}$, $F_{C2\beta}$, and $F_{C3\gamma}$) that satisfy this set of three equations is a solution that satisfies the static equilibrium and friction constraints.

To illustrate this process, we apply this strategy to the example system. Using the unit surface normal direction vectors u_{N1} , u_{N2} , and u_{N3} together with their corresponding friction coefficients μ_{C1} , μ_{C2} , and μ_{C3} , the equation for the friction cones and their friction constraint equations can be found as

$$Cone_{C1}(F_{C1\gamma}, F_{C1\alpha}, F_{C1\delta}) = 0.397F_{C1\gamma}^2 + 0.671F_{C1\alpha}^2 + 0.631F_{C1\delta}^2 + 0.019F_{C1\gamma}F_{C1\alpha} - 0.697F_{C1\alpha}F_{C1\delta} + 0.943F_{C1\delta}F_{C1\gamma} \le 0$$
(28)

$$Cone_{C2}(F_{C2\alpha}, F_{C2\beta}, F_{C2\beta}) = 0.492F_{C2\alpha}^{2} + 0.621F_{C2\beta}^{2} + 0.514F_{C2\beta}^{2}$$
$$- 0.029F_{C2\alpha}F_{C2\beta} - 0.859F_{C2\beta}F_{C2\beta}$$
$$+ 0.994F_{C2\beta}F_{C2\alpha} \leq 0$$
(29)

$$Cone_{C3}(F_{C3\beta}, F_{C3\gamma}, F_{C3\delta}) = 0.579F_{C3\beta}^2 + 0.571F_{C3\gamma}^2 + 0.508F_{C3\delta}^2 - 0.357F_{C3\beta}F_{C3\gamma} - 0.919F_{C3\gamma}F_{C3\delta} + 0.911F_{C3\delta}F_{C3\beta} \le 0$$
(30)

These friction cones are shown to scale in Figs. 3 and 6. Substituting $F_{C1\delta}$, $F_{C2\delta}$, and $F_{C3\delta}$ with the values already found [Eqs. (15)], we now have three friction constraint equations with two

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Fig. 6 Three friction cones sliced by their force planes

variables each in the form of conic section inequality equations as

$$\begin{split} E_{C1}^{\#}(F_{C1\gamma}F_{C1\alpha}) &= 0.397F_{C1\gamma}^{2} + 0.671F_{C1\alpha}^{2} + 0.019F_{C1\gamma}F_{C1\alpha} \\ &- 1.325F_{C1\alpha} + 1.792F_{C1\gamma} + 2.282 \leq 0 \quad (31) \end{split}$$

$$\begin{split} E^{\#}_{C2}(F_{C2\alpha},F_{C2\beta}) &= 0.492F^2_{C2\alpha} + 0.621F^2_{C2\beta} - 0.029F_{C2\alpha}F_{C2\beta} \\ &+ 1.866F_{C2\alpha} - 1.612F_{C2\beta} + 1.808 \leq 0 \quad (32) \end{split}$$

$$E_{C3}^{\#}(F_{C3\beta}, F_{C3\gamma}) = 0.579F_{C3\beta}^2 + 0.571F_{C3\gamma}^2 - 0.357F_{C3\beta}F_{C3\gamma} + 3.845F_{C3\beta} - 3.883F_{C3\gamma} + 9.057 \le 0$$
(33)

Geometrically, this can be understood as finding the cross section regions of the friction cones sliced by the force planes which are defined by the known force components ($F_{C1\delta}$, $F_{C2\delta}$, and $F_{C3\delta}$) that are parallel to the foot contact plane as shown in Fig. 6. Figure 7 shows the foot contact plane with these cross sections projected onto it. Each of these three conic section regions represents the region of the two force components which satisfy the friction constraint for their contact point.

To eliminate three variables from the three conic section inequality equations with six force component variables [Eqs. (31)–(33)], the three linear relationships [Eqs. (16)–(18)] are applied to each of them to obtain a new set of three conic section inequality equations with two variables each, but a total of only three force component variables $F_{C1\alpha}$, $F_{C2\beta}$, and $F_{C3\gamma}$ Geometrically, this process can be understood as applying simple linear transformations [Eqs. (16)–(18)] to the conic section regions ($E_{C1}^{\#}$, $E_{C2}^{\#}$, and $E_{C3}^{\#}$) to transform each of them to new regions (E_{C1} , E_{C2} , and E_{C3}) as shown in the transformation from Fig. 7 to Fig. 8. This transformation involves flipping and shifting of the conic sections in their nonorthogonal coordinate frames.

Applying the transformation given as the linear relationship Eq. (16) to the original conic section inequality equation $E_{C1}^{\#}$ as shown in Eq. (31), the new conic section inequality equation E_{C1} is obtained as

$$\begin{split} E_{C1}(F_{C3\gamma},F_{C1\alpha}) &= 0.397F_{C3\gamma}^2 + 0.671F_{C1\alpha}^2 - 0.019F_{C3\gamma}F_{C1\alpha} \\ &- 1.863F_{C3\gamma} - 1.323F_{C1\alpha} + 2.443 \leq 0 \quad (34) \end{split}$$

and applying the transformation Eq. (17) to $E_{C2}^{\#}$ as shown in Eq. (32), the new conic section inequality equation E_{C2} is obtained as



Fig. 7 Projection of the cross section regions onto the foot contact plane

$$\begin{split} E_{C2}(F_{C1\alpha},F_{C2\beta}) &= 0.492 F_{C1\alpha}^2 + 0.621 F_{C2\beta}^2 + 0.029 F_{C1\alpha} F_{C2\beta} \\ &\quad -1.818 F_{C1\alpha} - 1.610 F_{C2\beta} + 1.719 \leq 0 \quad (35) \end{split}$$

and finally, applying the transformation Eq. (18) to $E_{C3}^{\#}$ as shown in Eq. (33), the new conic section inequality equation E_{C3} is obtained as

$$E_{C3}(F_{C2\beta}, F_{C3\gamma}) = 0.579F_{C2\beta}^2 + 0.571F_{C3\gamma}^2 + 0.357F_{C2\beta}F_{C3\gamma} - 3.883F_{C2\beta} - 3.895F_{C3\gamma} + 9.184 \le 0$$
(36)



Fig. 8 Transformation of the projected cross section regions

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Fig. 9 The force space graph representation

Now we have a new set of three conic section inequality equations $(E_{C1}, E_{C2}, \text{ and } E_{C3})$ with a total of only three force component variables $F_{C1\alpha}$, $F_{C2\beta}$, and $F_{C3\gamma}$. Any set of forces that satisfy these newly formulated conic section inequality equations will satisfy all sets of static equilibrium and friction constraints.

3.6 Defining the Solution Space. Now that we have reduced the problem into a set of three quadratic inequality equations with three variables ($F_{C1\alpha}$, $F_{C2\beta}$, and $F_{C3\gamma}$), any set of forces that satisfy this set of three constraints is defined as the solution space. One way of representing this solution space graphically is to gather all three non orthogonal contact point coordinate axes on the foot contact plane such that all three contact points coincide at a single point as shown in Fig. 9. This representation helps to visualize the three quadratic inequality constraints and the solution space that satisfies them, and will also be used as a tool for choosing a solution in the solution space. We will call this representation the "force space graph."

Figure 10(*a*) shows an example of a set of three contact force components $F_{C1\alpha}$, $F_{C2\beta}$, and $F_{C3\gamma}$ on the force space graph as a valid solution where point P_1 , P_2 , and P_3 defined by these three contact force components are in each of their transformed conic sections, thus satisfying all constraints. Figure 10(*b*) shows an

example of an invalid solution on the force space graph where point P_1 is not inside its transformed conic section $E_{C1}(F_{C3\gamma}, F_{C1\alpha})$ and thus will slip at contact point C_1 .

Another way of representing the solution space graphically is by defining it as the volume created by the intersection of the projections of the three conic sections in three-dimensional space as shown in Fig. 11. We will call this representation the "solution volume." Any point defined by the three contact force components that is in this solution volume is a solution that satisfies all static equilibrium and friction constraints. The three conic sections used in generating the solution volume are from the same set of three conic section inequality equations E_{C1} , E_{C2} , and E_{C3} [Eqs. (34)–(36)]; however, since they are plotted on a three axis orthogonal system instead of on the nonorthogonal system used in the force space graph representation, the shapes of these conic sections are different from those in the force space graph representation.

The projection of the solution volume onto each plane describes the region of all possible solution points in each transformed conic section as shown in Fig. 11(b). The wider this region is for a transformed conic section the better the corresponding foot contact point, since this means that there are more choices for selecting a foot contact force for that particular foot. The shape and size of the solution volume and its projections onto each plane can provide useful information on the quality of the solution space for its particular system.

These two representations of the solution space (the force space graph and the solution volume representation) will give a better insight into the complex force interactions visually, help in choosing the optimal solution, and help see the quality of the chosen solution.

Using the strategy presented, we now have a representation of all the combinations of the force distributions possible that satisfy all static equilibrium and friction constraints as the entire solution space. Among the nine unknown force components, three of them are explicitly specified $(F_{C1\delta}, F_{C2\delta}, \text{and } F_{C3\delta})$, another three components $(F_{C1\alpha}, F_{C2\beta}, \text{and } F_{C3\gamma})$ are given as possible ranges and constraints by three quadratic inequality equations [Eqs. (25)–(27)] represented by the force space graph or the solution volume, and the last three components $(F_{C1\gamma}, F_{C2\alpha}, \text{ and } F_{C3\beta})$ are given as linear relationships to the other components [Eqs. (8)–(10)]. Now choosing a solution set is a matter of specifying the three contact force components variables $F_{C1\alpha}, F_{C2\beta}$, and $F_{C3\gamma}$ that satisfy the three quadratic inequality equations using the chosen criteria as shown in [1–3].



Fig. 10 Valid/invalid solution on the force space graph. (a) A valid solution; (b) An invalid solution (slip at C_1).

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Fig. 11 The solution volume representation. (a) The intersection of E_{C1}, E_{C2}, and E_{C3}; (b) The solution space as a volume.

4 Summary and Conclusion

In this paper, we have presented a method for finding the description of the contact force solution space and ways to visualize it for the three feet contact case with a simple example. Geometric properties of the friction cone and its conic section were used to find the solution space, and two representation schemes, the force space graph and the solution volume representation, were developed to describe the solution space and to be used as a visualization tool. Discussions for the geometric interpretation to the procedure and results were presented using an example to visually provide insight for the physical meanings of the parameters involved. The final solution may be chosen in this solution space using these representation schemes by selecting the values for the three parameters that define the solution using an appropriate criteria [1–3]. The strategy developed could be used for grasp planning of multi-fingered robotic hands as well.

Future research areas may include developing methods for systems with four elements: a method for the "two cable-two feet contact case" where the solution is described with two parameters, and for the "four feet contact case" where the solution is described with six parameters would be a natural extension to the current methods developed.

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