A Method for Representing the Configuration and Analyzing the Motion of Complex Cable-Pulley Systems

Dennis W. Hong
e-mail: dhong@ecn.purdue.edu

Raymond J. Cipra
e-mail: cipra@ecn.purdue.edu

School of Mechanical Engineering,
Purdue University
West Lafayette, IN 47907-2088

In this paper a systematic way of representing complex cable-pulley mechanism configurations and a method to analyze their motion is presented. This technique can also be used as an aid for synthesis. The cable-pulley system model that is being considered is planar and composed of three basic elements which are pulleys, blocks, and cables. A configuration table is used to identify the constraint equations by systematically defining the connections between the cables, pulleys, and blocks. The basic strategy is to use the constraint equations to generate the relationship between each variable and a subset of the variables identified as the inputs. A row reduction process on the system of constraint equations identifies the number of inputs and ultimately generates the relationships of each variable to the input(s). Results with different input variables can be easily obtained by a simple column interchange process. Examples are given to illustrate the procedure.
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1 Introduction

Flexible connectors such as belts, ropes, or chains are used for transmitting motion and power usually when shafts are so far apart that a gear drive would be inadvisable. They are also used as conveyors and hoists. The block and tackle arrangement has been used to gain a mechanical advantage in hoisting for a very long time. The Spanish Burton (Fig. 1(*a*)) and the Weston differential pulley block (Fig. 1(*b*)) are age-old, interesting examples. Cables with pulleys are also frequently used to generate parallel motion. Figure 1(*c*) shows one such arrangement often used to guide a straight edge on a drafting board. All of these mechanisms which use cables and pulleys have already been studied thoroughly. For their limited applications these mechanisms have simple configurations and thus have straightforward and simple kinematics often figured out by intuition.

However, cables and pulleys are now starting to be used in different configurations for new types of applications such as tendon-driven manipulators and cable-suspended robots and haptic interfaces. Cables can be considered as a ''link'' which when wrapped around a pulley form a ''joint'' connecting the cable and pulley in a nonslip kinematic relationship. By revisiting these simple mechanical elements (cables and pulleys), new types of complex mechanisms can be synthesized for new and exciting applications. However, a general method for analyzing such complex cable-pulley systems is not currently available.

In this paper we present a systematic way of representing complex cable-pulley mechanism configurations and a method to analyze their motion along with examples. Commercially available computer programs such as Working Model, ADAMS, or DADS can also handle certain complex cable pulley systems. However, the analysis strategy presented in this paper is much simpler, allows more control and provides deeper insight thus is invaluable for synthesis compared to such computer programs. One current limitation of the presented method is that the cables must be either in the X direction or the Y direction, and the orientation of the cables cannot change. A more general case where the cables can change their orientation is left as future work.

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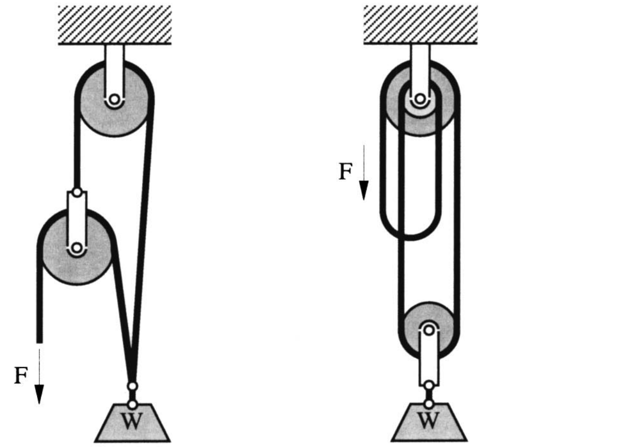
2 Previous Work

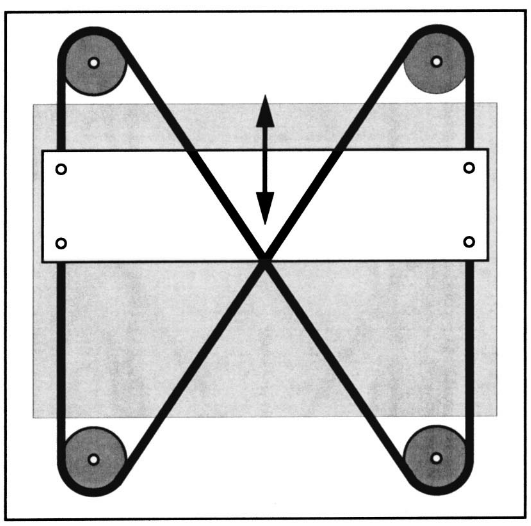
There are many advantages for using cables and pulleys in a mechanism. Structural simplicity, compactness, light weight, high stiffness, low friction, low backlash and the ability to absorb shock are some of the advantages which are especially important for robotics applications such as tendon-driven dexterous robotic hands. Some examples are the MIT/Utah hand [1] and the Salisbury hand [2] which use many different arrangements of cables and pulleys for their fingers. Tsai [3] gives an overview of the current state of the art in the design of tendon-driven manipulators which includes structure, classification, kinematics, statics, dynamics and control. Low cost and the ability to easily scale in size and produce large-scale mechanisms having a large workspace are also important advantages of using flexible connectors in a mechanism. Some interesting examples include Charlotte, a six degree of freedom tendon suspended platform robot developed for use on the Space Station [4], Robocrane, a six degree of freedom inverted Stewart platform which uses gravity to maintain tension in cables and was developed for use in shipping ports [5], and many different configurations of cable-suspended haptic interfaces (CSHI) such as the 4-cable CSHI [6], Texas 9-string [7] and the SPIDAR [8].

Tsai and Lee [9] investigate the kinematic structure of tendon driven robotic manipulators with the aid of graph theory. Using the concept of fundamental circuit, displacement equations of tendon-driven mechanisms are systematically derived from the kinematic structure. However, one of the assumptions in that work is that every pair of pulleys connected by a tendon must have a carrier in order to maintain a constant distance between the pulleys, thus the method cannot be used for certain mechanisms where the distance between pulleys can be changed, such as the block and tackle arrangement used in hoisting where the pulleys can move relative to each other.

Williams [10] addresses the issue of limited static workspace for cable-suspended robots and haptic interfaces due to the inability of the cable to exert compression by presenting the best design (with given constraints and parameters) for a planar 4-cable CSHI with a large static workspace via computer simulation. In order to solve this problem of limited static workspace in general, we wanted to synthesize a planar de-coupled CSHI such that all combinations of forces and moments are possible at all configurations

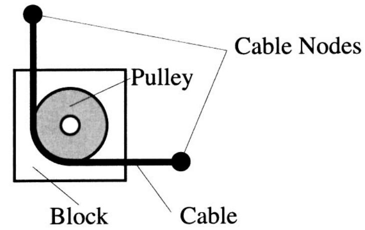


(a) Spanish Burton (b) Weston Differential Pulley Block

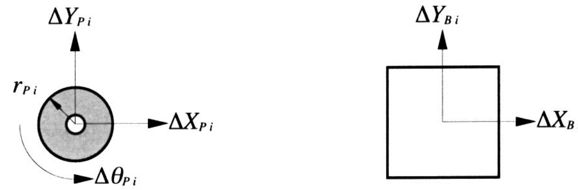


(c) Parallel Motion by Cables and Pulleys

Fig. 1 Examples of simple cable-pulley systems



(a) Basic Building Block



(b) Pulley P_i (c) Block B_i

Fig. 2 A basic building block with elements

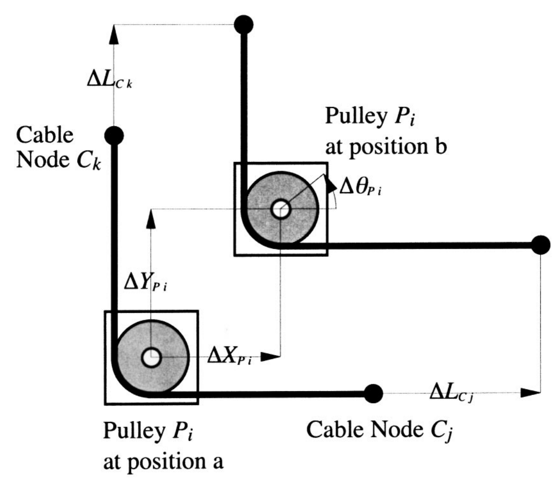
in its kinematic workspace. By attaching two cables on the two opposite sides of a block and maintaining them collinear by using pulleys and sliding blocks, the cables can now exert force on the block in the opposite two directions and thus eliminate actuation redundancy. However, a general method for analyzing such complex cable-pulley systems was not available. Thus, the motivation for coming up with this method of representing the configuration, and analyzing the motion of complex cable-pulley systems was to aid in synthesis.

3 Method and Procedure

3.1 Elements of a Cable-Pulley System. The cable-pulley system model that is being considered is planar and composed with three basic elements which are pulleys, blocks, and cables. A block at most can translate in two orthogonal directions. A pulley is attached to a block and can only rotate relative to the block. A cable can wrap around a pulley without slip and has a given constraint with respect to the pulley. The cables are either in the X direction or the Y direction, and the orientation of the cables do not change. Any point on a cable segment between two block or pulley elements is a potential cable node. The basic building block which contains a pulley, a block, and two cable nodes of the cable wrapped around the pulley is shown in Fig. 2(a). Each element is defined by, and operates according to the following rules:

(a) ***Pulley (P_i)***

- Three variables ΔX_{Pi}, ΔY_{Pi} and $\Delta\theta_{Pi}$, measured in the ground coordinate frame, represent the change in position in the X direction and the Y direction of the center of the pulley i, and the change in rotation angle of pulley i respectively (Fig. 2(b)).
- The constant r_{Pi} is the radius of the pulley i (Fig. 2(b)).
- Pulleys are attached to a block and thus their XY motions are constrained by the XY motion of that block (for pulley i attached to block k: $\Delta X_{Pi} = \Delta X_{Bk}$, $\Delta Y_{Pi} = \Delta Y_{Bk}$).
- A cable does not slip over the pulley, thus the rotation and the X or Y motion of the pulley together constrains the motion of



$$\Delta L_{Cj} = \Delta X_{Pi} + r_{Pi}\,\Delta\theta_{Pi}$$
$$\Delta L_{Ck} = \Delta Y_{Pi} - r_{Pi}\,\Delta\theta_{Pi}$$

Fig. 3 Change in position of cable nodes

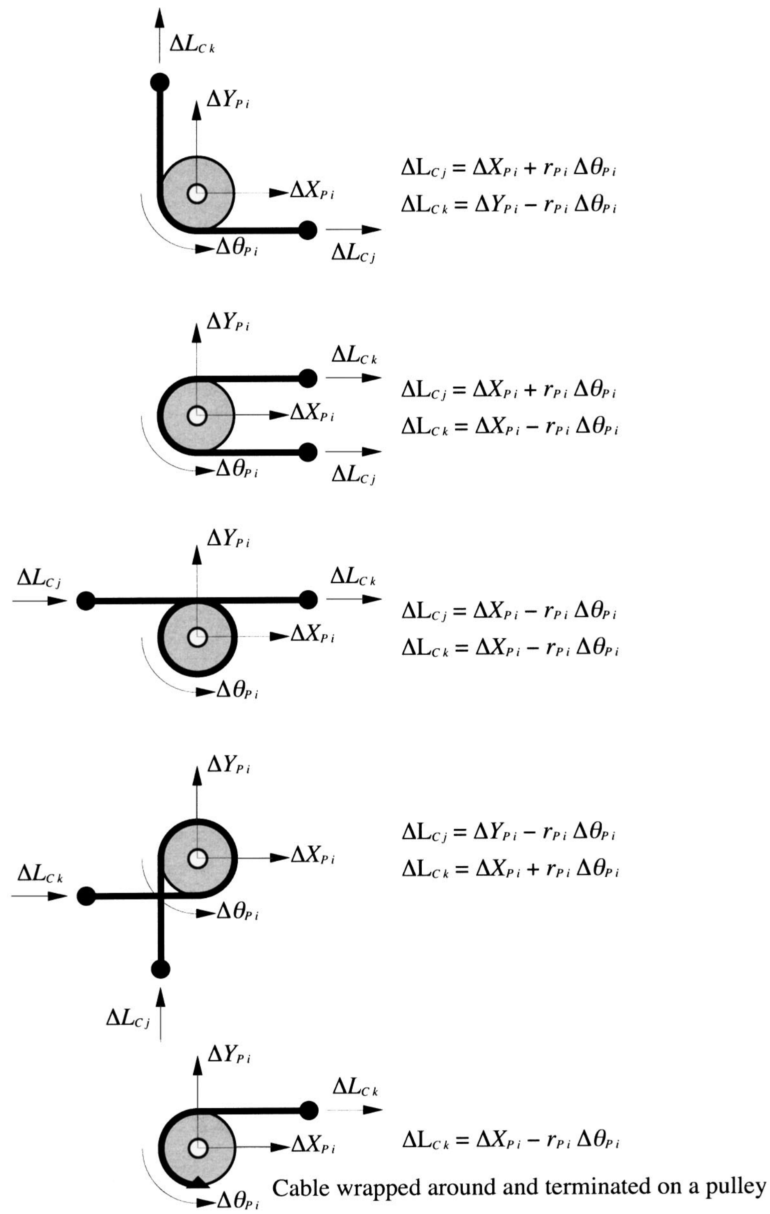


Fig. 4 Cable winding examples

the cable node in that cable direction. Thus ΔL_{Cj}, the change in position of the cable node C_j in the direction of the cable, is given by $\Delta L_{Cj} = \Delta X_{Pi} \pm r_{Pi} \Delta\theta_{Pi}$ for a cable in the X direction, or $\Delta L_{Cj} = \Delta Y_{Pi} \pm r_{Pi} \Delta\theta_{Pi}$ for a cable in the Y direction where the $\pm$ signs are determined by inspection. An example is shown in Fig. 3.

- Depending on how the cable is wound over a pulley, the rotation contribution to the change in motion of the cable node in the cable direction can either increase or decrease for a positive direction rotation of the pulley. Figure 4 shows some examples where inspection was used to generate the ΔL_C equation.

(*b*) ***Block (B_i)***

- Two variables ΔX_{Bi} and ΔY_{Bi} measured in the ground coordinate frame, represent the change in position of block i in the X direction and the Y direction respectively (Fig. 2(*c*))
- Blocks can have sliding constraints relative to other blocks ($\Delta X_{Bj} = \Delta X_{Bk}$ for sliding in the Y direction or $\Delta Y_{Bi} = \Delta Y_{Bj}$ for sliding in the X direction) or can be stationary as ground ($\Delta X_{Bi} = 0$, $\Delta Y_{Bi} = 0$) (Fig. 5)
- Blocks can not rotate.

(*c*) ***Cable Node (C_i)***

- A cable node is defined as a point on a cable between two block or pulley elements (between two pulleys, between two blocks, or between a block and a pulley).
- ΔL_{Ci} is the change in position of the cable node i in that cable direction only (Fig. 3). Even if the cable node changes its position in other directions as well, we are only interested in the position change of that cable node in its cable direction.
- A cable can be a closed loop (endless cable) or an open loop (open-ended cable).

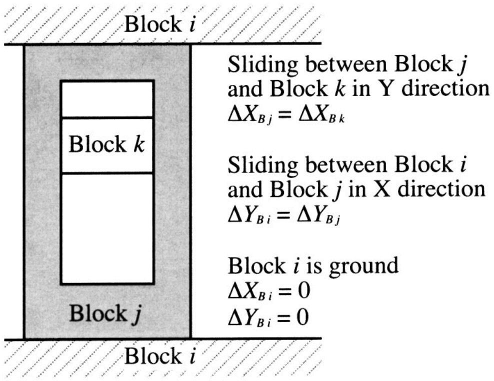


Fig. 5 Sliding constraints between blocks

- Both ends of an open-ended cable must be constrained in motion by being connected to a block or wrapped around and terminated on a pulley (Fig. 4).
- The cable can not stretch and thus the total length of a cable does not change.
- The orientation of the cable is either in the X direction or in the Y direction and does not change.
- Cables are always in tension and thus there is no slack.

3.2 Analysis Method. The basic strategy is to use the constraint equations to generate the relationship between each variable and a subset of the variables identified as the inputs. This is done by first systematically identifying all of the constraint equations, then choosing the input variables, the number of which are equal to the total number of variables minus the number of independent constraints. This choosing of the set of inputs is done in conjunction with a row reduction process on the system of constraint equations which identifies the number of inputs and ultimately generates the relationship of each variable to the input(s).

The identification of the constraint equations is simplified by the use of a configuration table which shows the topology of the system and defines the connection between the cables, pulleys, and blocks. The table is set up with the blocks (B_i) and pulleys (P_i) making up the columns, and the blocks (B_i) and cable nodes (C_i) making up the rows as shown for example in Table 1. The entries in the table then provide specific relationships between the elements of the corresponding row and column.

As can be seen, there are four subsections of the table corresponding to four categories of constraint equations between particular elements (block-block, block-pulley, cable-block, cablepulley). Within each subsection the following entries identify constraints.

(*a*) Block-Block Subsection (indicate constraints between blocks)

- $-$ sliding in X direction (constrained in Y direction)
- $|$ sliding in Y direction (constrained in X direction)
- $+$ no relative motion (only used on the diagonal of the block-block subsection indicating the constraint between the same block)

Table 1 Format of a configuration table

	B_1	B_2	B_3	P_1	P_2
B_1					
B_2					
B_3					
C_1					
C_2					
C_3					

(*b*) Block-Pulley Subsection (indicate constraints between a block and a pulley)

- • pulley attached to block

(*c*) Cable-Block Subsection (indicate constraints between a cable node and a block)

- Y cable attached to block in Y direction
- X cable attached to block in X direction

(*d*) Cable-Pulley Subsection (indicate constraints between a cable node and a pulley)

- $X+$ cable wound over pulley in X direction such that L_C increases as θ_P increases ($\Delta L_{Cj} = \Delta X_{Pi} + r_{Pi}\Delta\theta_{Pi}$)
- $X-$ cable wound over pulley in X direction such that L_C decreases as θ_P increases ($\Delta L_{Cj} = \Delta X_{Pi} - r_{Pi}\Delta\theta_{Pi}$)
- $Y+$ cable wound over pulley in Y direction such that L_C increases as θ_P increases ($\Delta L_{Cj} = \Delta Y_{Pi} + r_{Pi}\Delta\theta_{Pi}$)
- $Y-$ cable wound over pulley in Y direction such that L_C decreases as θ_P increases ($\Delta L_{Cj} = \Delta Y_{Pi} - r_{Pi}\Delta\theta_{Pi}$)

The number of constraint equations can be calculated from the configuration table by the following procedure. Sum the number of entries above the diagonal in the block-block subsection and add two times the number of entries in the block-pulley subsection and add the number of entries in the cable-block and cable-pulley subsections and lastly add 2 for the block chosen to represent ground. The number of variables can be calculated by summing two times the numbers of blocks and three times the number of pulleys along with the number of cable nodes.

All of the constraint equations identified from the configuration table are put into a matrix equation form $\mathbf{Ax} = \mathbf{0}$ where $\mathbf{A}$ is the system matrix and $\mathbf{x}$ is the column vector with all of the variables. The system matrix $\mathbf{A}$ is then row reduced to generate the row reduced system matrix $\mathbf{U}$. Now the matrix equation $\mathbf{Ux} = \mathbf{0}$ is used to identify any dependent constraints, to determine the number of inputs needed, and to generate relationships between each variable and the inputs.

The inputs or outputs can be any of the variables (ΔX_{Pi}, ΔY_{Pi}, $\Delta\theta_{Pi}$, ΔX_{Bi}, ΔY_{Bi} or ΔL_{Ci}) of any element (any i). However, the number of input variables must be equal to the degree of freedom of the system and they must be independent. The choice of input variables can be aided by looking at the row reduced system matrix $\mathbf{U}$ and observing the degree of freedom of the system and the dependencies between the variables.

The number of dependent constraints in the total constraint equations is the number of zero rows in the row reduced system matrix $\mathbf{U}$. Thus the degree of freedom of the system is the number of non-zero columns not counting the columns of the identity matrix portion in the row reduced system matrix $\mathbf{U}$. The variables in the column vector $\mathbf{x}$ which correspond to these non-zero columns in the row reduced system matrix $\mathbf{U}$ are chosen as the input variables. Thus it is desirable to move the columns corresponding to the variables which are desired to be used as the inputs to the last columns in the system matrix $\mathbf{A}$ before beginning the row reduction process. The following example (Example 2) shows this procedure in detail.

Among the output variables, the variables for stationary elements (constants) are not interesting since their change in position is always zero. We are not interested in the pulley position variables either since pulleys are always connected to blocks, and thus the change in position of the pulleys are always identical to those of the blocks they are attached to. The rest of the output variables are the ‘‘variables of interest’’ (nonconstant, nonredundant output variables) and we choose to show only the relationship between these ‘‘variables of interest’’ and the input variables as the results.

The procedure to analyze cable pulley systems can be summarized as follows:

- Number the blocks, pulleys, and cable nodes.
- Generate the configuration table based on the connections

(sliding constraints between blocks, pulleys attached to blocks, cables attached to blocks in which direction, cable is wound over pulley.)
- Using the configuration table write constraint equations between blocks, between blocks and pulleys, between cable nodes and blocks, and between cable nodes and pulleys.
- Choose a ground block.
- Put the constraint equations into matrix form $\mathbf{Ax}=\mathbf{0}$ (system matrix $\mathbf{A}$)
- Choose potential input variables and move the columns corresponding to these input variables to the last columns in the system matrix $\mathbf{A}$.
- Row reduce system matrix $\mathbf{A}$ (row reduced system matrix $\mathbf{U}$) to determine the number of independent constraints and the number of inputs.
- Use the row reduced system matrix $\mathbf{U}$ to identify output variable relationships to input variables.

4 Examples

To demonstrate the proposed method, three cable-pulley mechanisms are analyzed.

4.1 Example 1. For this example, a single degree of freedom mechanism with an open-ended cable which only moves in the Y direction is considered as shown in Fig. 6. The mechanism is simple enough to figure out the motion relationship between the elements by inspection. The mechanism is composed of two pulleys, three blocks and a single open-ended cable. Block 1 is the ground and block 3 is considered as input. As the input block is pulled down, it can be easily seen that block 2 will move up half the amount.

Following the procedure, the configuration table is constructed as shown in Table 2. Note that the upper left part of the configuration table (block-block subsection) which shows the sliding constraint relationship between the blocks is always symmetric, and it's diagonal elements are always a "+".

From the configuration table, the constraint equations are set up. For this mechanism, there are 14 equations and 15 variables. There are 2 equations to describe the ground block, and the rest of the equations correspond to the four subsections of the configuration table respectively.

(*a*) Ground Block Equations

$$\Delta X_{B1}=0$$

$$\Delta Y_{B1}=0$$

(*b*) Block Sliding Constraint Equations

$$\Delta X_{B2}=\Delta X_{B1}$$

$$\Delta X_{B3}=\Delta X_{B1}$$

(*c*) Pulley-Block Attachment Constraint Equations

$$\Delta X_{P1}=\Delta X_{B2}$$

$$\Delta Y_{P1}=\Delta Y_{B2}$$

$$\Delta X_{P2}=\Delta X_{B1}$$

$$\Delta Y_{P2}=\Delta Y_{B1}$$

(*d*) Cable-Block Connection Constraint Equations

$$\Delta L_{C1}=\Delta Y_{B1}$$

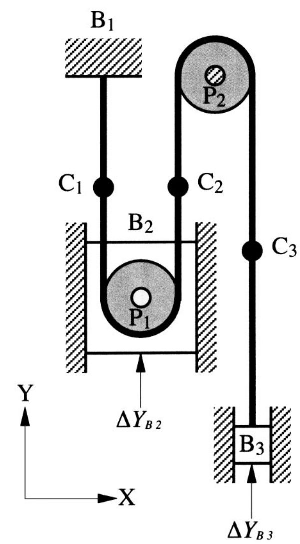


Fig. 6 Example 1

Table 2 Configuration table for example 1

	B_1	B_2	B_3	P_1	P_2
B_1	+	|	|		•
B_2	|	+		•	
B_3	|		+		
C_1	Y			Y-	
C_2				Y+	Y-
C_3			Y		Y+

$$\Delta L_{C3}=\Delta Y_{B3}$$

(*e*) Cable Node-Pulley Constraint Equations

$$\Delta L_{C1}=\Delta Y_{P1}-r_{P1}\Delta\theta_{P1}$$

$$\Delta L_{C2}=\Delta Y_{P1}+r_{P1}\Delta\theta_{P1}$$

$$\Delta L_{C2}=\Delta Y_{P2}-r_{P2}\Delta\theta_{P2}$$

$$\Delta L_{C3}=\Delta Y_{P2}+r_{P2}\Delta\theta_{P2}$$

The 14 constraint equations with the 15 variables are put into a matrix equation form of

$$\mathbf{Ax}=\mathbf{0} \tag{1}$$

where $\mathbf{A}$ is the 14 by 15 system matrix and $\mathbf{x}$ is the column vector with the 15 variables. The order of the columns in the system matrix is arbitrary; however, it is convenient to put the column corresponding to the chosen input variable, in this case ΔY_{B3} as the last column. The degrees of freedom of the system is the dimension of the row space of the system matrix (number of columns of $\mathbf{A}-$rank of $\mathbf{A}$) or the number of non-zero columns not counting the columns of the identity matrix portion in the row reduced system matrix $\mathbf{U}$. Writing out the matrix $\mathbf{A}$, Eq. (1) becomes:

$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&-1&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&-1&0&0&0&1&0\\ 0&0&0&1&0&0&0&0&0&0&0&-1&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&1&0&0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&-1&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&-1\\ 1&0&0&0&-1&r_{p1}&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&-1&-r_{p1}&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&-1&r_{p2}&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&-1&-r_{p2}&0&0&0&0&0&0 \end{bmatrix} \begin{Bmatrix} \Delta L_{C1}\\ \Delta L_{C2}\\ \Delta L_{C3}\\ \Delta X_{P1}\\ \Delta Y_{P1}\\ \Delta\theta_{P1}\\ \Delta X_{P2}\\ \Delta Y_{P2}\\ \Delta\theta_{P2}\\ \Delta X_{B1}\\ \Delta Y_{B1}\\ \Delta X_{B2}\\ \Delta Y_{B2}\\ \Delta X_{B3}\\ \Delta Y_{B3} \end{Bmatrix} = \begin{Bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{Bmatrix} \tag{2}$$

The system matrix **A** is row reduced to an echelon form **U**, and thus

$$\mathbf{Ux}=\mathbf{0} \tag{3}$$

where:

$$\mathbf{U}=\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&-1\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&\frac{1}{2}\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&\frac{1}{2r_{p1}}\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&\frac{-1}{r_{p2}}\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&\frac{1}{2}\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \end{bmatrix} \tag{4}$$

Now the displacement equations of the variables with respect to the input variable can be obtained.

The dimension of the row space, or the number of nonzero columns not counting the columns of the identity matrix portion in the row reduced system matrix **U** is one, thus the degree of freedom of the system is one as expected. With the input variable chosen as ΔY_{B3}, the displacement equations for the variables of interest are shown as the following:

(*a*) Block Position

$$\Delta Y_{B2}=(-1/2)\Delta Y_{B3}$$

$$\Delta Y_{B3}=\text{input}$$

(*b*) Pulley Rotation

$$\Delta\theta_{P1}=(-1/2r_{P1})\Delta Y_{B3}$$

$$\Delta\theta_{P2}=(1/r_{P2})\Delta Y_{B3}$$

(*c*) Cable Node Position

$$\Delta L_{C2}=-\Delta Y_{B3}$$

$$\Delta L_{C3}=\Delta Y_{B3}$$

4.2 Example 2. For the second example, a single degree of freedom mechanism with an open-ended cable which moves both in the X and Y direction is considered as shown in Fig. 7. The mechanism is simple enough to figure out the motion relationship between the elements by inspection. The mechanism is composed of two pulleys, three blocks and a single open-ended cable with block 1 as the ground. First, the position of block 3 in the Y

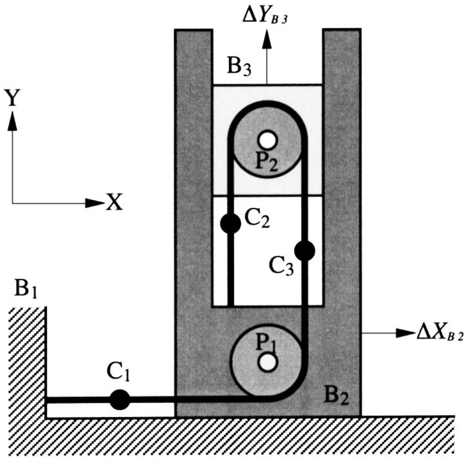


Fig. 7 Example 2

direction is considered as input. As the input (block 3) is pulled up, it can be easily seen that block 2 will move to the left twice the amount. Then, another variable will be chosen as input to demonstrate the column interchange strategy for obtaining the relationships between the variables and various inputs.

Following the procedure, the configuration table is constructed as shown in Table 3. Note that block 1 and block 2 can slide in the X direction relative to each other. Also cable node 1 is connected to block 1 in the X direction, and wraps around pulley 1 in the X direction.

The constraint equations then are set up from the configuration table. For this mechanism, there are 14 equations and 15 variables. There are 2 equations to describe the ground block, and the rest of the equations correspond to the four subsections of the configuration table respectively.

The 14 constraint equations with the 15 variables are put into a matrix equation form $\mathbf{Ax}=\mathbf{0}$ where $\mathbf{A}$ is the 14 by 15 system matrix and $\mathbf{x}$ is the column vector with the 15 variables. The system matrix $\mathbf{A}$ is then row reduced to an echelon form $\mathbf{U}$, and now the displacement equations of the variables with respect to the desired input variable can be obtained. The dimension of the row space, or the number of nonzero columns not counting the columns of the identity matrix portion in the row reduced system matrix $\mathbf{U}$ is one, thus the degree of freedom of the system is one as expected. The matrix equation with the row reduced system matrix $\mathbf{U}$ with ΔY_{B3} as the input is shown in Eq. (5). Note that the chosen input variable ΔY_{B3} is the last element in the column vector $\mathbf{x}$ and the column corresponding to this variable is the nonzero column (last column) in the row reduced system matrix $\mathbf{U}$.

$$\left[\begin{array}{c|c} I & c_1 \end{array}\right]\left\{\begin{array}{c} \vdots \\ \Delta Y_{B3} \end{array}\right\}=0 \tag{5}$$

With the input variable chosen as ΔY_{B3}, from the matrix equation with the row reduced system matrix (Eq. (5)), the displacement equations for the variables of interest are as shown:

Table 3 Configuration table for example 2

	B_1	B_2	B_3	P_1	P_2
B_1	+	–			
B_2	–	+	|	•	
B_3		|	+		•
C_1	X			X+	
C_2		Y			Y-
C_3				Y+	Y+

$$\Delta X_{B2}=(-2)\Delta Y_{B3}$$
$$\Delta X_{B3}=(-2)\Delta Y_{B3}$$
$$\Delta Y_{B3}=\text{input}$$
$$\Delta\theta_{P1}=(2/r_{P1})\Delta Y_{B3}$$
$$\Delta\theta_{P2}=(1/r_{P2})\Delta Y_{B3}$$
$$\Delta L_{C3}=2\Delta Y_{B3}$$

If the displacement equations with a different input variable are needed, the system matrix is rearranged such that the column corresponding to the new input variable becomes the last column in the new system matrix. The matrix equation with the row reduced system matrix $\mathbf{U}$ with $\Delta\theta_{P2}$ chosen as input is shown in Eq. (6).

$$\left[\begin{array}{c|c} I & c_3 \end{array}\right]\left\{\begin{array}{c} \vdots \\ \Delta\theta_{P2} \end{array}\right\}=0 \tag{6}$$

From this, the displacement equations for the variables of interest are shown as the following:

$$\Delta X_{B2}=(-2)r_{P2}\Delta\theta_{P2}$$
$$\Delta X_{B3}=(-2)r_{P2}\Delta\theta_{P2}$$
$$\Delta Y_{B3}=r_{P2}\Delta\theta_{P2}$$
$$\Delta\theta_{P1}=(2r_{P2}/r_{P1})\Delta\theta_{P2}$$
$$\Delta\theta_{P2}=\text{input}$$
$$\Delta L_{C3}=2r_{P2}\Delta\theta_{P2}$$

These sets of results for different input variables can be obtained from one another by simply manipulating the set of equations for any chosen input variable. However, as the mechanism gets complex and equations coupled, obtaining results for different input variables by simple equation manipulation becomes difficult as will be shown in the third example. Thus the column interchange method shown is preferred for obtaining displacement equations for different variables.

4.3 Example 3. The last example shown in Fig. 8 is a closed loop (endless cable) mechanism with 10 pulleys and 3 blocks where block 3 can be moved both in the X and Y direction independently. It is quite complicated such that it is not so easy to figure out the relationship between the variables by inspection. It is a three degree of freedom system (as will be shown by the method later) and the choice of the allowable three independent input variables is not so obvious. The twist in the second pulley is

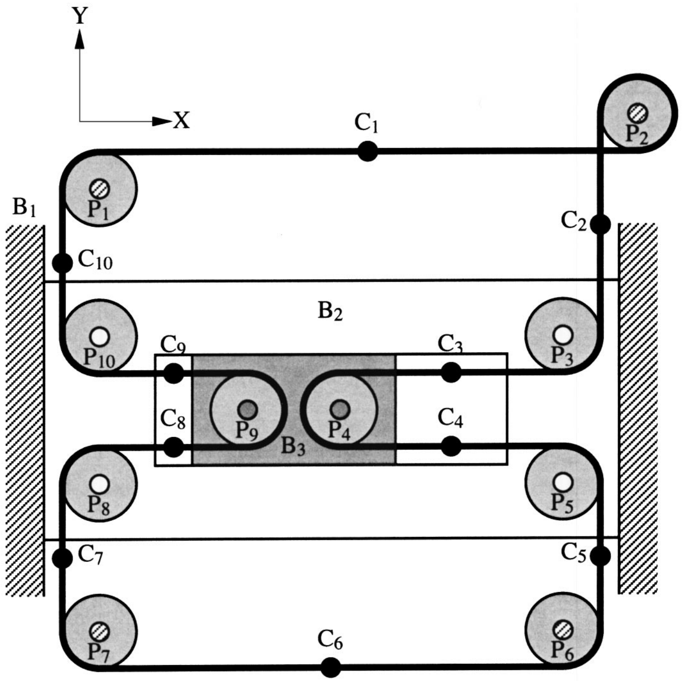


Fig. 8 Example 3

just to demonstrate one of the different cable winding possibilities. This mechanism might be used as a base for a three degree of freedom planar positioning device or a CSHI.

Following the procedure, the configuration table is constructed as shown in Table 4. Note that since it is a closed loop (endless cable), there are no cable-block connections. Also note that 4 pulleys are attached to block 1, 4 pulleys are attached to block 2, and 2 pulleys are attached to block 3 and thus their motions are constrained by the motion of that block respectively. From the configuration table, the constraint equations are set up. There are 2 equations to describe the ground block, and the rest of the equations correspond to the four subsections of the configuration table respectably.

(*a*) Ground Block Equations

$$\Delta X_{B1}=0$$

$$\Delta Y_{B1}=0$$

(*b*) Block Sliding Constraint Equations

$$\Delta X_{B2}=\Delta X_{B1}$$

$$\Delta Y_{B3}=\Delta Y_{B2}$$

(*c*) Pulley-Block Attachment Constraint Equations
Pulleys P_1, P_2, P_6 and P_7 are constrained to block B_1 (8 equations).
Pulleys P_3, P_5, P_8 and P_{10} are constrained to block B_2 (8 equations).
Pulleys P_4 and P_9 are constrained to block B_3 (4 equations).

(*d*) Cable-Block Connection Constraint Equations
No equations since it is a closed loop (endless cable).

(*e*) Cable Node-Pulley Constraint Equations

$$\Delta L_{C1}=\Delta X_{P1}-r_{P1}\Delta\theta_{P1}$$

$$\Delta L_{C1}=\Delta X_{P2}+r_{P2}\Delta\theta_{P2}$$

$$\Delta L_{C2}=\Delta Y_{P2}-r_{P2}\Delta\theta_{P2}$$

Table 4 Configuration table for example 3

	B_1	B_2	B_3	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
B_1	+	$\vert$		•	•				•	•			
B_2	$\vert$	+	–			•		•			•		•
B_3		–	+				•					•	
C_1				X-	X+								
C_2					Y-	Y+							
C_3						X+	X-						
C_4							X+	X-					
C_5								Y+	Y+				
C_6									X+	X+			
C_7										Y-	Y-		
C_8											X-	X+	
C_9												X-	X+
C_{10}				Y-									Y-

$$\Delta L_{C2}=\Delta Y_{P3}+r_{P3}\Delta\theta_{P3}$$

$$\Delta L_{C3}=\Delta X_{P3}+r_{P3}\Delta\theta_{P3}$$

$$\Delta L_{C3}=\Delta X_{P4}-r_{P4}\Delta\theta_{P4}$$

$$\Delta L_{C4}=\Delta X_{P4}+r_{P4}\Delta\theta_{P4}$$

$$\Delta L_{C4}=\Delta X_{P5}-r_{P5}\Delta\theta_{P5}$$

$$\Delta L_{C5}=\Delta Y_{P5}+r_{P5}\Delta\theta_{P5}$$

$$\Delta L_{C5}=\Delta Y_{P6}+r_{P6}\Delta\theta_{P6}$$

$$\Delta L_{C6}=\Delta X_{P6}+r_{P6}\Delta\theta_{P6}$$

$$\Delta L_{C6}=\Delta X_{P7}+r_{P7}\Delta\theta_{P7}$$

$$\Delta L_{C7}=\Delta Y_{P7}-r_{P7}\Delta\theta_{P7}$$

$$\Delta L_{C7}=\Delta Y_{P8}-r_{P8}\Delta\theta_{P8}$$

$$\Delta L_{C8}=\Delta X_{P8}-r_{P8}\Delta\theta_{P8}$$

$$\Delta L_{C8}=\Delta X_{P9}+r_{P9}\Delta\theta_{P9}$$

$$\Delta L_{C9}=\Delta X_{P9}-r_{P9}\Delta\theta_{P9}$$

$$\Delta L_{C9}=\Delta X_{P10}+r_{P10}\Delta\theta_{P10}$$

$$\Delta L_{C10}=\Delta Y_{P10}-r_{P10}\Delta\theta_{P10}$$

$$\Delta L_{C10}=\Delta Y_{P1}-r_{P1}\Delta\theta_{P1}$$

The 44 constraint equations with the 46 variables are put into a matrix equation form where **A** is the 44 by 46 system matrix and **x** is the column vector with the 46 variables; however, the system's degrees of freedom is not two. Note that the last row in the system matrix **A** becomes all zeroes after row reduction thus indicating a dependent constraint equation exists and resulting in a three degree of freedom system as shown in Eq. (7). If the desired inputs are known in advance, the system matrix is rearranged such that the columns corresponding to the desired input variables become the last columns in the system matrix. The system matrix **A** is then row reduced to an echelon form **U**, and now the displacement equations of the variables with respect to the desired input variables can be obtained from columns C_1, C_2, and C_3 as shown in Eq. (7).

$$\boldsymbol{U}=\left[\begin{array}{c|ccc} \boldsymbol{I} & c_1 & c_2 & c_3 \\ \hline & \mathbf{0} & & \end{array}\right] \tag{7}$$

4.3.1 Results. The following results are with the default input variables $\Delta\theta_{P10}$, ΔX_{B3}, ΔY_{B3}. Due to the complexity of the mechanism, some of the results shown below are not so obvious and are difficult to derive by inspection.

(*a*) Block Position

$$\Delta Y_{B2}=\Delta Y_{B3}$$

$$\Delta X_{B3}=\text{input } B$$

$$\Delta Y_{B3}=\text{input } A$$

(*b*) Pulley Rotation

$$\Delta\theta_{P1}=(r_{P10}/r_{P1})\Delta\theta_{P10}+(-1/r_{P1})\Delta Y_{B3}$$

$$\Delta\theta_{P2}=(-r_{P10}/r_{P2})\Delta\theta_{P10}+(1/r_{P2})\Delta Y_{B3}$$

$$\Delta\theta_{P3}=(r_{P10}/r_{P3})\Delta\theta_{P10}+(-2/r_{P3})\Delta Y_{B3}$$

$$\Delta\theta_{P4}=(-r_{P10}/r_{P4})\Delta\theta_{P10}+(1/r_{P4})\Delta X_{B3}+(2/r_{P4})\Delta Y_{B3}$$

$$\Delta\theta_{P5}=(r_{P10}/r_{P5})\Delta\theta_{P10}+(-2/r_{P5})\Delta X_{B3}+(-2/r_{P5})\Delta Y_{B3}$$

$$\Delta\theta_{P6}=(r_{P10}/r_{P6})\Delta\theta_{P10}+(-2/r_{P6})\Delta X_{B3}+(-1/r_{P6})\Delta Y_{B3}$$

$$\Delta\theta_{P7}=(r_{P10}/r_{P7})\Delta\theta_{P10}+(-2/r_{P7})\Delta X_{B3}+(-1/r_{P7})\Delta Y_{B3}$$

$$\Delta\theta_{P8}=(r_{P10}/r_{P8})\Delta\theta_{P10}+(-2/r_{P8})\Delta X_{B3}$$

$$\Delta\theta_{P9}=(-r_{P10}/r_{P9})\Delta\theta_{P10}+(1/r_{P9})\Delta X_{B3}$$

$$\Delta\theta_{P10}=\text{input } C$$

(*c*) Cable Node Position

$$\Delta L_{C1}=-r_{P10}\Delta\theta_{P10}+\Delta Y_{B3}$$

$$\Delta L_{C2}=r_{P10}\Delta\theta_{P10}-\Delta Y_{B3}$$

$$\Delta L_{C3}=r_{P10}\Delta\theta_{P10}-2\Delta Y_{B3}$$

$$\Delta L_{C4}=-r_{P10}\Delta\theta_{P10}+2\Delta X_{B3}+2\Delta Y_{B3}$$

$$\Delta L_{C5}=r_{P10}\Delta\theta_{P10}-2\Delta X_{B3}-\Delta Y_{B3}$$

$$\Delta L_{C6}=r_{P10}\Delta\theta_{P10}-2\Delta X_{B3}-\Delta Y_{B3}$$

$$\Delta L_{C7}=-r_{P10}\Delta\theta_{P10}+2\Delta X_{B3}+\Delta Y_{B3}$$

$$\Delta L_{C8}=-r_{P10}\Delta\theta_{P10}+2\Delta X_{B3}$$

$$\Delta L_{C9}=r_{P10}\Delta\theta_{P10}$$

$$\Delta L_{C10}=-r_{P10}\Delta\theta_{P10}+\Delta Y_{B3}$$

4.3.2 Results (as the Forward Kinematic Solution of a Planar Robot). If one is interested in this mechanism as a planar positioning table or a cable suspended robot, the three input variables could be $\Delta\theta_{P1}$, $\Delta\theta_{P7}$, $\Delta\theta_{P10}$ in pulleys 1, 7, and 10, for example, and the output would be the position of block 3 (ΔX_{B3},ΔY_{B3}) and the orientation of either pulley ($\Delta\theta_{P4}$ or $\Delta\theta_{P9}$) on that block.

The system matrix **A** is rearranged such that the three columns corresponding to the new input variables $\Delta\theta_{P1}$, $\Delta\theta_{P7}$, $\Delta\theta_{P10}$ become the last three columns in the new system matrix. The system matrix is then row reduced to an echelon form **U**, and the displacement equations of the variables of interest with respect to these new input variables are obtained.

(*a*) Block Position

$$\Delta Y_{B2}=-r_{P1}\Delta\theta_{P1}+r_{P10}\Delta\theta_{P10}$$

$$\Delta X_{B3}=(r_{P1}/2)\Delta\theta_{P1}-(r_{P7}/2)\Delta\theta_{P7}$$

$$\Delta Y_{B3}=-r_{P1}\Delta\theta_{P1}+r_{P10}\Delta\theta_{P10}$$

(*b*) Pulley Rotation

$$\Delta\theta_{P1}=\text{input } A$$

$$\Delta\theta_{P2}=(-r_{P1}/r_{P2})\Delta\theta_{P1}$$

$$\Delta\theta_{P3}=(2r_{P1}/r_{P3})\Delta\theta_{P1}+(-r_{P10}/r_{P3})\Delta\theta_{P10}$$

$$\Delta\theta_{P4}=(-3r_{P1}/2r_{P4})\Delta\theta_{P1}-(r_{P7}/2r_{P4})\Delta\theta_{P7}+(r_{P10}/r_{P4})\Delta\theta_{P10}$$

$$\Delta\theta_{P5}=(r_{P1}/r_{P5})\Delta\theta_{P1}+(r_{P7}/r_{P5})\Delta\theta_{P7}-(r_{P10}/r_{P5})\Delta\theta_{P10}$$

$$\Delta\theta_{P6}=(r_{P7}/r_{P6})\Delta\theta_{P7}$$

$$\Delta\theta_{P7}=\text{input } B$$

$$\Delta\theta_{P8}=(-r_{P1}/r_{P8})\Delta\theta_{P1}+(r_{P7}/r_{P8})\Delta\theta_{P7}+(r_{P10}/r_{P8})\Delta\theta_{P10}$$

$$\Delta\theta_{P9}=(r_{P1}/2r_{P9})\Delta\theta_{P1}-(r_{P7}/2r_{P9})\Delta\theta_{P7}-(r_{P10}/r_{P9})\Delta\theta_{P10}$$

$$\Delta\theta_{P10} = \text{input} \quad C$$

4.3.3 Results (as the Inverse Kinematic Solution of a Planar Robot). If one is interested in this mechanism as a cable suspended robot, the three input variables could be the position of block 3 ($\Delta X_{B3}, \Delta Y_{B3}$) and the orientation of pulley 4 ($\Delta\theta_{P4}$) and the 3 output variables $\Delta\theta_{P1}$, $\Delta\theta_{P7}$, $\Delta\theta_{P10}$ could be solved for as the three variables for the actuated joints for example. Note that it is difficult to obtain the following set of results (with $\Delta\theta_{P4}$, ΔX_{B3}, ΔY_{B3} as the input) by simple equation manipulation of the forward kinematic solution results (with $\Delta\theta_{P1}$, $\Delta\theta_{P7}$, $\Delta\theta_{P10}$ as the input) shown previously.

(*a*) Block Position

$$\Delta Y_{B2} = \Delta Y_{B3}$$

$$\Delta X_{B3} = \text{input} \quad B$$

$$\Delta Y_{B3} = \text{input} \quad A$$

(*b*) Pulley Rotation

$$\Delta\theta_{P1} = (-r_{P4}/r_{P1})\Delta\theta_{P4} + (1/r_{P1})\Delta X_{B3} + (1/r_{P1})\Delta Y_{B3}$$

$$\Delta\theta_{P2} = (r_{P4}/r_{P2})\Delta\theta_{P4} + (-1/r_{P2})\Delta X_{B3} + (-1/r_{P2})\Delta Y_{B3}$$

$$\Delta\theta_{P3} = (-r_{P4}/r_{P3})\Delta\theta_{P4} + (1/r_{P3})\Delta X_{B3}$$

$$\Delta\theta_{P4} = \text{input} \quad C$$

$$\Delta\theta_{P5} = (-r_{P4}/r_{P5})\Delta\theta_{P4} + (-1/r_{P5})\Delta X_{B3}$$

$$\Delta\theta_{P6} = (-r_{P4}/r_{P6})\Delta\theta_{P4} + (-1/r_{P6})\Delta X_{B3} + (1/r_{P6})\Delta Y_{B3}$$

$$\Delta\theta_{P7} = (-r_{P4}/r_{P7})\Delta\theta_{P4} + (-1/r_{P7})\Delta X_{B3} + (1/r_{P7})\Delta Y_{B3}$$

$$\Delta\theta_{P8} = (-r_{P4}/r_{P8})\Delta\theta_{P4} + (-1/r_{P8})\Delta X_{B3} + (2/r_{P8})\Delta Y_{B3}$$

$$\Delta\theta_{P9} = (r_{P4}/r_{P9})\Delta\theta_{P4} + (-2/r_{P9})\Delta Y_{B3}$$

$$\Delta\theta_{P10} = (-r_{P4}/r_{P10})\Delta\theta_{P4} + (1/r_{P10})\Delta X_{B3} + (2/r_{P10})\Delta Y_{B3}$$

5 Conclusion

A systematic way of representing complex cable-pulley mechanism configurations and a method to analyze their motion is presented in this paper. First, a configuration table is used to show the topology of the system and to identify the constraint equations by systematically defining the connections between the elements (cables, pulleys, and blocks). These constraint equations are then put into a matrix equation form called the system matrix. A row reduction process on this system matrix identifies the number of inputs and generates the relationships of each variable to the input(s). The number of input variables or the degrees of freedom of the system is the dimension of the row space of the system matrix. Results with different set of input variables can be easily obtained by a simple column interchange process on the system matrix. This simple analysis method provides insight to the motion of complex cable-pulley systems and thus is invaluable for the synthesis of new types of complex cable-pulley configurations for new and exciting applications. However, the method can only handle cases when the cables are either in the *X* direction or the *Y* direction, and the orientation of the cables do not change. A more general case where the cables can change their orientation is left as future work.

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